Unsteady Pseudoplastic Flow Near a Moving Wall

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There are available in the literature (*1, 2, 3*) analytical solutions for the power-law (Ostwald-de Waele) model for pseudoplastic non-Newtonian fluids for a number of different geometries. These solutions are however all for steady flow. Here the solution to an unsteady flow problem is given.

A semi-infinite body of pseudoplastic fluid extends from $y = 0$ to $y = \infty$ and is bounded on one side by a solid surface imbedded in the xz plane. Initially the fluid is at rest, but for time $t \geq 0$ the solid surface moves in the x direction with a constant velocity. It is desired to know the x component of the velocity v_x as a function of the distance from the solid surface and the time for a pseudoplastic fluid which obeys the following relation for the components of the momentum flux:

$$\tau_{ij} = -m\left(\tfrac{1}{2} \sum_k \sum_l \Delta_{kl}^2\right)^{(n-1)/2} \Delta_{ij} \qquad (1)$$

In this expression $\Delta_{ij} = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)$ is the ij component of the rate-of-strain tensor.* For the problem stated above this relationship gives

$$\tau_{xy} = -m\,|\partial v_x/\partial y|^{n-1}\,(\partial v_x/\partial y) \qquad (2)$$

for the xy component of the momentum flux.

The equation of motion for the system is

$$\rho(\partial v_x/\partial t) = -(\partial \tau_{xy}/\partial y) \qquad (3)$$

Substituting Equation (2) into Equation (3), taking into account the fact that $\partial v_x/\partial y$ is everywhere negative, one obtains the partial differential equation for the velocity distribution:

$$\rho(\partial v_x/\partial t) = -m\frac{\partial}{\partial y}(-\partial v_x/\partial y)^n \qquad (4)$$

This is to be solved with the boundary conditions $v_x = V$ at $y = 0$, $v_x = 0$ at $y = \infty$, and the initial condition that $v_x = 0$ for $t < 0$. According to the method of combination of variables y and t are combined into a new dimensionless variable:

$$r = (n+1)^{-1}y(\rho/mtV^{n-1})^{1/(n+1)} \qquad (5)$$

Now one anticipates that the dimensionless velocity $\phi_n = v_x/V$ will be a function of r alone. This allows Equation (4) to be transformed into the following ordinary differential equation:

$$\phi_n''(-\phi_n')^{n-1} + (n+1)^n n^{-1} r\phi_n' = 0 \qquad (6)$$

with boundary conditions such that $\phi_n = 1$ at $r = 0$, and $\phi_n = 0$ at $r = \infty$; primes denote differentiation with respect to r. For $n = 1$ (Newtonian flow) Equation (6) becomes $\phi_1'' + 2r\phi_1' = 0$, for which the solution (*6*) is $\phi_1 = \operatorname{erfc} r$; here $r = y/\sqrt{4\mu t/\rho}$.

For $n < 1$ (pseudoplastic flow) Equation (6) may be integrated twice to give

$$\phi_n = \beta_n^{-1/(1-n)} \cdot \int_r^\infty (B_n + r^2)^{-1/(1-n)}\,dr \qquad (7)$$

in which $\beta_n = (1+n)^n(1-n)/2n$. The constant of integration B_n is determined from the boundary condition at $r = 0$, which may be written as

$$1 = \beta_n^{-1/(1-n)} \cdot \int_0^\infty (B_n + r^2)^{-1/(1-n)}\,dr \qquad (8)$$

If now a new variable defined by

$$s^2 = r^2/(B_n + r^2) \qquad (9)$$

is introduced, then integrals of the following form appear in Equations (7) and (8):

$$I_n(s) = \int_s^1 (1-u^2)^{(3n-1)/2(1-n)}\,du \qquad (10)$$

In terms of these quantities the integration constant and the dimensionless velocity distribution assume the final form

$$B_n = \beta_n^{-2/(1+n)}[I_n(0)]^{2(1-n)/(1+n)} \qquad (11)$$

$$\phi_n = I_n(s)/I_n(0) \qquad (12)$$

(*Continued on page 6D*)

TABLE 1. NUMERICAL VALUES FOR CALCULATING VELOCITY PROFILES

n	$I_n(0)$	B_n	r_1
1/3	1	0.8660	6.57
1/2	$\pi/4$	1.6370	4.30
2/3	$3\pi/16$	2.8383	3.05
5/6	$63\pi/512$	5.9782	2.29
1	. . .	. . .	1.83

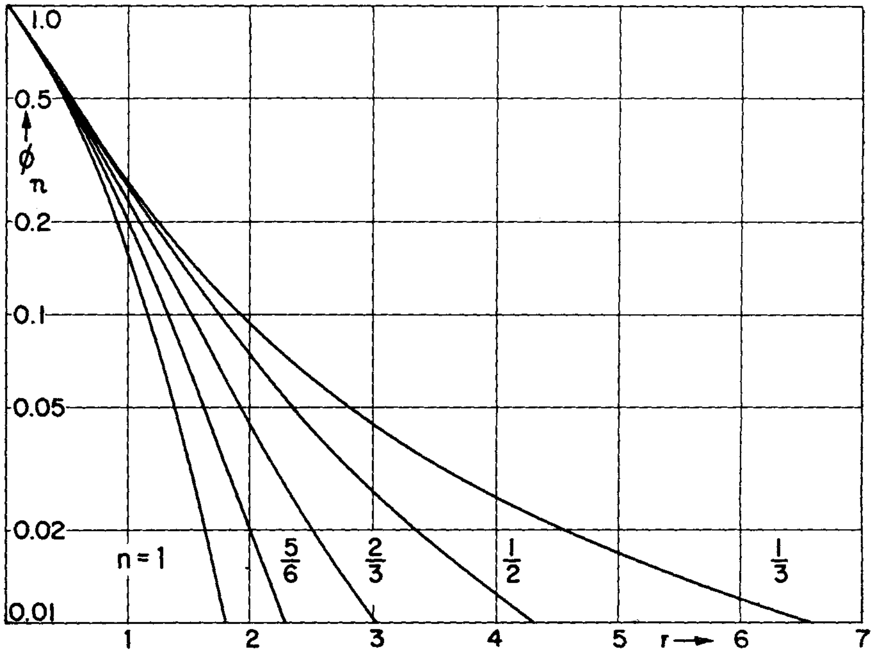

Fig. 1. Dimensionless velocity profiles for flow of a pseudoplastic fluid near a flat surface suddenly set in motion with a constant velocity.

*The writing of the power law in the form given in Equation (1) insures proper behavior under coordinate transformation (*4*). Equation (3) in a previous paper (*5*) should have been written in this way; clearly the unspecified form used there was adequate for dimensional considerations.

(Continued from page 565.)

In the accompanying table numerical values needed for calculating several velocity profiles are given; the equations for the profiles are

$$\phi_{1/3} = 1 - s \qquad (13)$$

$$\phi_{1/2} = 1 - (2/\pi)(cs + \arcsin s) \qquad (14)$$

$$\phi_{2/3} = 1 - (2/\pi)(\tfrac{2}{3}c^3s + cs + \arcsin s) \qquad (15)$$

$$\phi_{5/6} = 1 - (2/\pi)\Big(\frac{128}{315}c^9s + \frac{16}{35}c^7s + \frac{8}{15}c^5s + \tfrac{2}{3}c^3s + cs + \arcsin s\Big) \qquad (16)$$

wherein $c = (1 - s^2)^{1/2}$. These curves are presented in the accompanying figure. From the curves one can obtain the values of the reduced variable r for which the fluid velocity has fallen off to 1% of the velocity of the moving wall; this value r_1 given in the table is a measure of the extent of momentum penetration.

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NOTATION

B_n = constant of integration
m, n = parameters characterizing a pseudoplastic fluid
r = reduced variable
s = variable defined in Equation (9)
t = time after wall begins to move
V = constant velocity of moving wall
y = distance from moving wall
μ = Newtonian viscosity
ρ = fluid density
ϕ_n = dimensionless velocity distribution

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